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ABSTRACT

Expressions are presented for the electric and magnetic fields due to a pulse of charge, which may be oscillating transversely while moving down an infinitely long highly conducting pipe of circular cross section. The expressions are evaluated at large distances from the pulse and the fields are shown to decrease algebraically in the distance behind the pulse. In the absence of transverse oscillations the longitudinal electric field varies as the inverse three-halves power of the distance; in the presence of oscillations the dominant field component is the transverse magnetic field, which decreases as the inverse one-half power. In the long-range limit the amplitude of the fields is proportional to the square root of the wall resistivity. The phase of the field associated with the oscillating pulse is shown to be the phase of the pulse at the time when it passed the point of observation.

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INTRODUCTION

It has been shown that the finite conductivity of the walls of an accelerator vacuum chamber can lead to unstable coherent oscillations of azimuthally uniform beams.[1,2] The question of stability arises for a longitudinally bunched beam in which the distance between bunches is large compared with the radius of the vacuum pipe. If the electric and magnetic fields fall off fast enough with distance from the bunch, the motion of separate bunches would be independent of one another. It has been shown that the local self fields of a bunch do not lead to unstable motion.[3] Therefore one might expect to stabilize coherent beam oscillations by bunching the beam longitudinally.

If the vacuum chamber walls are infinitely conducting, the fields fall off exponentially in a distance of the order of the pipe radius (which is typically small compared with the distance between bunches), and therefore a longitudinal bunching of a uniform beam *would* stabilize the coherent motion.

It is the purpose of this paper to obtain expressions for the fields at large distances from a moving bunch of charge surrounded by walls with finite conductivity. These fields are the basic ingredients in an analysis of the coherent motion of a bunched beam.[4] We limit our analysis to obtaining the fields at distances large compared with the pipe radius and the bunch length. The conductivity of the wall is such that the displacement current in the wall can be neglected compared with the conduction current.

A discussion is given of the dependence of the fields upon distance from the pulse, with particular attention to the different functional dependences which occur at various distances. Over a very large range the fields are shown to fall off algebraically, and in agreement with the independent results of a number of workers;[5,6] the most important aspect of the work reported here is a careful delineation of the range of validity of these previously obtained formulas.

The important results for the analysis of the coherent motion of azimuthally bunched beams are that: (i), the dominant term in the longitudinal force of one bunch on a subsequent bunch decreases algebraically with the distance z between bunches as $|z|^{-3/2}$ (Eq. 1.16a); and (ii), the dominant term in the transverse force has a phase that depends only upon position (as measured in the laboratory), and an amplitude that decreases algebraically with distances between bunches as $|z|^{-1/2}$ [Eq. (2.20)]. Suffice it to say, here, that bunched beams are not generally stable and the stability criteria are different from that for uniform beams. Discussion of all of this may be found in Ref. 4 and forthcoming papers based on the abstracts of Ref. 4.

In the first section, the fields created by rectilinear longitudinal motion of a pulse of charge are obtained; in the second section the fields created by transverse oscillation of the pulse are derived.

The general mathematical method which we employ, namely the use of Fourier transforms, was suggested by S. Weinberg's analysis[7] of a related problem. In Appendix A we discuss some mathematical questions associated with approximating Fourier integrals, and summarize the transforms employed in this paper. Appendix B summarizes properties of Bessel functions which are required in the analysis.

I. PURELY LONGITUDINAL MOTION

1. Derivation of the Fields

In this section we obtain the expressions for the electric and magnetic fields arising from a bunch of charge in purely longitudinal motion. The pulse of charge moves in the z direction with velocity v inside an infinitely long straight pipe of circular cross section and wall conductivity σ. The inner and outer radii of the pipe are b and d, respectively. The pulse of charge has constant radial density inside a radius a.. The charge and current density are taken as

$$\rho_0(r,z,t) = n_0\, f(z-v\,t)\, H(a-r)\ , \tag{1.1a}$$

$$J_{0z}(r,z,t) = v\,\rho_0(r,z,t)\ , \tag{1.1b}$$

$$J_{0y} = J_{0x} = 0\ , \tag{1.1c}$$

where cylindrical coordinates are used, and $H(x)$ is the Heaviside unit step function that is unity for positive argument and zero for negative argument. The function $f(x)$ is normalized such that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1\ . \tag{1.2}$$

Consequently $\pi a^2 n_0 = eN$, with N the number of particles in the pulse.

Due to the symmetry of ρ_0 and J_0, only E_z, E_r, and B_θ are nonzero. It will be useful to use Fourier transformations in solving for the fields, and the convention will be adopted that a tilde above a quantity designates the transform as defined by

$$f(z - v\,t) = \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{i\,k(z - v\,t)}\, dk\,, \tag{1.3a}$$

$$E_z(z - v\,t) = \int_{-\infty}^{\infty} \tilde{E}_z(k)\, e^{i\,k(z - v\,t)}\, dk\,. \tag{1.3b}$$

analogous expressions hold for E_r and B_θ.

We define the following regions:

Region 1	$0 < r < a$,
Region 2	$a < r < b$,
Region 3	$b < r < d$,
Region 4	$d < r$.

From Maxwell's equations and Ohm's law we obtain the relationships between the field components in the various regions. In regions 1,2, and 4 we have

$$\tilde{E}_r = \frac{ik}{q^2}\,\frac{\partial \tilde{E}_z}{\partial r}\,, \tag{1.4a}$$

$$\tilde{B}_\theta = \frac{ik\beta}{q^2}\,\frac{\partial \tilde{E}_z}{\partial r}\,, \tag{1.4b}$$

with $$q^2 = -\,(k/\gamma)^2 \quad \text{and} \quad \gamma^2 = [1 - (v/c)^2]^{-1}\,.$$

In region 3 (inside the metal) we have

$$\tilde{E}_r = \frac{ik}{\alpha^2} \frac{\partial \tilde{E}_z}{\partial r} , \tag{1.5a}$$

$$\tilde{B}_\theta = \frac{(ik\beta - 4\pi\sigma/c)}{\alpha^2} \frac{\partial \tilde{E}_z}{\partial r} , \tag{1.5b}$$

where $\alpha^2 = q^2 + Rik$ and $R = 4\pi\beta\sigma/c$.

In region 1, the equation for $\tilde{E}_z$ takes the form

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{E}_z}{\partial r} \right) + q^2 \tilde{E}_z = i \frac{4\pi n_0 \tilde{f}(k)}{\gamma^2} k . \tag{1.6a}$$

In regions 2 and 4 we have

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{E}_z}{\partial r} \right) + q^2 \tilde{E}_z = 0 , \tag{1.6b}$$

and in region 3

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{E}_z}{\partial r} \right) + \alpha^2 \tilde{E}_z = 0 . \tag{1.6c}$$

Equations (1.6) are zero-order Bessel equations, or simply related to Bessel's equation. Various properties of the solution to Bessel's equation, that are used in this work, are given in Appendix B .

At this point it is necessary to clarify our definition of q and α . So far, only q^2 and α^2 have been defined and we are left with a choice of sign for q and α . The sign convention is arbitrary, however, it is convenient to choose the sign in such a way that the imaginary part of q and α is always positive.

The expressions for E_z that are finite everywhere are

$$\text{Region 1:}\quad \tilde{E}_z = C_1 J_0(qr) + \frac{4\pi n_0 \tilde{f}}{ik} , \tag{1.7a}$$

$$\text{Region 2:}\quad \tilde{E}_z = \left[C_1 - \frac{2\pi^2 n_0 q a \tilde{f}}{ik} N_1(qa)\right] J_0(qr) + \left[\frac{2\pi^2 n_0 q a \tilde{f}}{ik} J_1(qa)\right] N_0(qr) , \tag{1.7b}$$

$$\text{Region 3:}\quad \tilde{E}_z = C_3 J_0(\alpha r) + C_4 N_0(\alpha r) , \tag{1.7c}$$

$$\text{Region 4:}\quad \tilde{E}_z = C_5 H_0^{(1)} (qr) . \tag{1.7d}$$

In these expressions $J_n(x)$, $N_n(x)$ and $H_n^{(1)}(x)$ are defined, in Appendix B , by Eqs. (B.2), (B.3), and (B.4). The constants C_i , $i = 1,3,4,5$ are to be determined from the continuity conditions of the fields at the boundaries between the various regions. The boundary conditions at $r = a$ are already satisfied by the expressions for $\tilde{E}_z$, while the continuity of $\tilde{E}_z$ and $\tilde{B}_\theta$ at $r = b$ gives

$$C_3 = -\frac{\pi}{2}\left\{\left[C_1 - \frac{2\pi^2 n_0 q\, a\tilde{f}}{ik} N_1(qa)\right]\left[\alpha b\, J_0(qb) N_1(\alpha b) - \frac{i\,k\,\alpha^2\, b\,\beta}{q(ik\beta - 4\pi\sigma/c)} \times J_1(qb)\, N_0(\alpha b)\right] + \left[\frac{2\pi^2 n_0 q\, a\tilde{f}}{ik} J_1(qa)\right]\left[\alpha b N_0(qb)\, N_1(\alpha b) - \frac{i\,k\,\alpha^2\, b\,\beta}{q(ik\beta - 4\pi\sigma/c)} N_1(qb)\, N_0(\alpha b)\right]\right\} , \tag{1.8a}$$

and

$$C_4 = \frac{\pi}{2}\left\{\left[C_1 - \frac{2\pi^2 n_0 q a \tilde{f}}{ik} N_1(qa)\right]\left[\alpha b\, J_0(qb)\, J_1(\alpha b) - \frac{ik\,\alpha^2 b\,\beta}{q(ik\beta - 4\pi\sigma/c)}\right.\right.$$

$$\times\; J_1(qb)\, J_0(\alpha b)\Big] + \left[\frac{2\pi^2 n_0 q\, a\tilde{f}}{ik} J_1(qa)\right]\Big[\alpha b\, N_0(qb)\, J_1(\alpha b)$$

$$\left.\left. - \frac{ik\,\alpha^2 b\,\beta}{q(ik\beta - 4\pi\sigma/c)} N_1(qb)\, J_0(\alpha b)\right]\right\} , \qquad (1.8b)$$

where we have employed Eq. (B.8). The boundary conditions at $r = d$ are used to obtain

$$- C_3\left[\alpha d\, J_1(\alpha d)\, H_0^{(1)}(qd) - \frac{ik\beta\,\alpha^2 d}{q(ik\beta - 4\pi\sigma/c)} J_0(\alpha d)\, H_1^{(1)}(qd)\right]$$

$$= C_4\left[\alpha d\, N_1(\alpha d) H_0^{(1)}(qd) - \frac{ik\beta\,\alpha^2 d}{q(ik\beta - 4\pi\sigma/c)} N_0(\alpha d) H_1^{(1)}(qd)\right] . \qquad (1.9)$$

Equations (1.8) and (1.9) are solved for C_1 , C_3 , and C_4 , and these constants are then substituted into Eqs. (1.7) to obtain the expression for $\tilde{E}_z$. When this general expression is inserted into Eq. (1.3b) the resulting Fourier integral is much too complicated to be performed analytically. Numerical integration would yield the complete expression for $\tilde{E}_z(z - vt)$ valid for all values of z and t . In the following subsection we shall restrict ourselves to large values of $(z - vt)$, and present results of the integration for three simplified geometries.

2. Field Expressions for Certain Geometries

(i) Wall removed ($b \to \infty$ or $\sigma \to 0$)

In this example the fields are those of a pulse of charge in free space so that $C_4 = i\, C_3$. On the cylinder axis $(r = 0)$, Eq. (1.7a) yields

$$\tilde{E}_z(k) = \frac{4\pi\, n_0 \tilde{f}}{ik} \left[1 - i\, \frac{\pi}{2}\; qa\; H_1^{(1)}(qa)\right] , \tag{1.10}$$

where use has been made of Eqs. (B.9) and (B.10) to simplify the coefficient C_1 . Equation (1.10) is to be substituted into the Fourier integral Eq. (1.3b) in order to determine $\tilde{E}_z(z - vt)$. For the region $|z - vt| \gg a$ we may invoke Appendix A - Case 2 to conclude that $ka \ll 1$; consequently we may employ the approximation of Eq. (B.5) and obtain

$$\tilde{E}_z(k) \approx i\, n_0 a^2\, k\, \ell n\, |ka| . \tag{1.11}$$

In obtaining this expression we have made use of the fact that, for $|z - vt|$ large compared with the length of the bunch, the field is independent of the form of $f(z - vt)$ and we may replace $f(k)$ by $f(0) = 1/2\pi$.

We now use the results of Table I (Appendix A) to obtain

$$E_z(z - vt) = \frac{eN}{\gamma^2} \; \frac{\mathrm{Sign}(z - vt)}{(z - vt)^2} , \tag{1.12}$$

where N is the number of particles in the bunch. This is the result one would obtain from a more elementary treatment; the factor γ^{-2} originates in the Lorentz contract. [The same result could be obtained, directly from Eq. (1.10), by invoking Appendix A - Case 1.]

(ii) Wall of infinite thickness ($d \to \infty$)

In this example an infinitely thick conducting wall surrounds the pulse. Because of the relativistic velocity of the beam and the high conductivity of the wall, conduction-current terms dominate inside the metal. Consequently, for example, $\alpha^2 \approx R\, i\, k$, so that

$$\frac{i\, k\, \beta\, \alpha^2}{q(ik\beta - 4\pi\sigma/c)} \approx -\beta^2 \gamma^2 q \quad . \tag{1.13}$$

With this approximation and taking $d \to \infty$ we obtain in the region $r < b$

$$\tilde{E}(k) = i\, \frac{n_0\, a^2\, \beta^2\, k}{\alpha\, b}\, \frac{H_0^{(1)}(\alpha b)}{H_1^{(1)}(\alpha b)} \quad , \tag{1.14}$$

where we have again restricted ourselves to a distance $|z - vt|$ much greater than both the vacuum chamber's inner radius and the pulse length, so that (by Appendix A - Case 2) $k\, b \ll 1$ and $\tilde{f} \approx 1/2\pi$.
In obtaining Eq. (1.14) we have employed Eqs. (B.2), (B.3), and (B.4).

Even with this restriction there are two regions of interest. The first is the region in which $|z - vt| \ll R\, b^2$ ($R \approx 10^8$ cm.$^{-1}$ for copper) and the second is the region in which $|z - vt| \gg R\, b^2$.

For $|z - vt| \ll R\,b^2$ we employ Appendix A - Case 3, noting that most of the contribution to the Fourier integral [Eq. (1.3b)] occurs for values of $\alpha\, b \gg 1$; for $|z - vt| \gg R\, b^2$ we have the situation of Appendix A - Case 1 , and may take $\alpha\, b \ll 1$. We use these facts, along with the proper approximations from Appendix B , to obtain the following approximate expressions for $\tilde{E}_z(k)$:

$$\tilde{E}_z(k) = i \frac{eN\beta^2}{\pi b(2R)^{\frac{1}{2}}} |k|^{\frac{1}{2}} [i + \mathrm{Sign}(k)], \quad |z - vt| \ll Rb^2 , \tag{1.15a}$$

$$\tilde{E}_z(k) = -i \frac{eN\beta^2}{2\pi} k\, [\ell n|kb| - i \tfrac{\pi}{2} \mathrm{Sign}(k)], \quad |z - vt| \gg Rb^2 , \tag{1.15b}$$

where a term proportional to k has been omitted in Eq. (1.15b) since it contributes to $E_z(z,t)$ only in the region of the pulse. We now use the results of Table I (Appendix A) to obtain

$$E_z(z,t) = \frac{eN\beta^2}{(\pi R)^{\frac{1}{2}}\, b} \frac{S(s,t)}{|s|^{3/2}} , \qquad |z - vt| \ll Rb^2 , \tag{1.16a}$$

$$E_z(z,t) = eN\beta^2 \frac{S(z,t)}{s^2} , \qquad |z - vt| \gg Rb^2 , \tag{1.16b}$$

where $s = (z - vt)$ and $S(z,t)$ is defined as zero for $z > vt$ and unity for $z < vt$. Thus we see that the field at large distance from the pulse is zero in front of the pulse but falls off algebraically behind the bunch. Equation (1.16) presents only the term with the slowest falloff and completely ignores the fields with a falloff distance of the order of the pipe's inner radius or the pulse length.

(iii) Thin wall $[(d - b) < b]$.

In order to simplify the algebra we will restrict ourselves, in this example, to a bunch that fills the pipe, and to an observation point at the pipe radius. Thus we take $r = a = b$. [Actually, as is suggested by Eqs. (1.16), we expect our results to be valid even without these restrictions, but we have not studied the more general case.] As with examples (ii) we ignore the displacement current and obtain

$$\tilde{E}_z(k) = i \frac{eN\beta^2 k}{\pi b}$$

$$\times \left\{ \frac{\alpha d \ \ell n \ |kd| [N_0(\alpha b) \ J_1(\alpha d) - J_0(\alpha b) \ N_1(\alpha d)]}{\alpha d \ \ell n \ |kd| [N_1(\alpha b) J_1(\alpha d) - J_1(\alpha b) N_1(\alpha d)] - \alpha\beta^2\gamma^2 [N_1(\alpha b) J_0(\alpha d) - J_1(\alpha b) N_0(\alpha d)]} \right\} \tag{1.17}$$

where again we have restricted ourselves to large distances, so that $|z - vt| \gg d$, and have consequently (Appendix A - Case 2) used the expansions of Eq. (B.2) and (B.3) that are valid for $kd \ll 1$.

The electric field $E_z(z,t)$ can be obtained by evaluating the Fourier inversion [Eq. (1.3b)] with Eq. (1.17) for $\tilde{E}_z(k)$.

We will restrict our attention, here, to two regions in which the integral can be readily approximated. The first region, $|z - vt| \ll R(d - b)^2$, has the major contribution to the Fourier integral occurring for values of $\alpha(d - b) \gg 1$, so $\tilde{E}_z(k)$ can be approximated by (Again, Appendix A - Case 3)

$$E_z(k) \;=\; i \;\; \frac{e\,N\,\beta^2}{\pi\,b\,(2R)^{1/2}} \;\; |k|^{1/2}\,[i + \mathrm{Sign}(k)] \;\; , \qquad (1.18)$$

where use has been made of Eq. (B.6) and the fact that

$$\frac{\cos[(i \pm 1)x]}{\sin[(i \pm 1)x]} \;\approx\; - i$$

for large x . The second region considered is $|z - vt| \gg Rd^2$; by Appendix A - Case 1 we may take $\alpha d^2 \ll 1$, and $E_z(k)$ is approximated by

$$\widetilde{E}_z(k) \;=\; \frac{i\;eNk}{\pi\;\gamma^2}\;\ell n\;|kd| \;\; , \qquad (1.19)$$

where use has been made of Eqs. (B.2) and (B.3). We now use Table I, to obtain

$$E(z,t) \;=\; \frac{e\,N\,\beta^2}{(\pi R)^{1/2}\;b}\;\;\frac{S(z,t)}{|s|^{3/2}} \;\; , \quad |z - vt| \ll R(d - b)^2, \qquad (1.20)$$

and

$$E(z,t) \;=\; \frac{eN}{\gamma^2}\;\;\frac{\mathrm{Sign}(z - vt)}{s^2}\;, \quad |z - vt| \gg Rd^2 \; . \qquad (1.21)$$

II. TRANSVERSE OSCILLATIONS WITH UNIFORM LONGITUDINAL MOTION

1. Exact Formulas for the Fields

In this section we solve for the electric and magentic field due to a pulse of charge oscillating transversely, in the x direction with amplitude ξ and frequency ω, while traveling longitudinally, in the z direction with constant velocity v. As in the preceding section, the charge is surrounded by an infinitely long straight pipe with circular cross section, conductivity σ, and inner radius b. The outer wall radius is taken to be infinite.

The amplitude ξ is assumed small compared to the beam radius a so that we may take the charge and current distribution to be

$$\rho = \rho_0 + \rho_1 \tag{2.1a}$$

and

$$\underset{\sim}{J} = \underset{\sim}{J}_0 + \underset{\sim}{J}_1 \quad , \tag{2.1b}$$

where ρ_0 and $\underset{\sim}{J}_0$ are defined by Eq. (1.1), and

$$\rho_1(r,z,t) = n_0 \, \xi \cos\theta \, \delta(r - a) \, f(z - vt) \, e^{-i\omega t} \quad , \tag{2.2a}$$

$$\underset{\sim}{J}_1 = n_0 \, \xi \, f(z - vt) \, e^{-i\omega t} \left\{ i\omega \, H(a - r) \left[- \cos\theta \, \hat{\underset{\sim}{e}}_r + \sin\theta \, \hat{\underset{\sim}{e}}_\theta \right] \right.$$

$$\left. + \; v \, \delta(r - a) \cos\theta \, \hat{\underset{\sim}{e}}_z \right\} \quad , \tag{2.2b}$$

with $H(x)$ the Heaviside step function, and $\delta(x)$ the Dirac delta function.

The field due to the sources ρ_0 and $\underset{\sim}{J}_0$ has been presented in Section I ; we will consider only the fields due to the sources ρ_1 and $\underset{\sim}{J}_1$ in this section. The total field will, of course, be the superposition of the fields due to each set of sources.

Again it will be useful to use Fourier transformations in solving for the fields. The definition, Eq. (1.3a), is still valid for $\tilde{f}(k)$, but we shall replace Eq. (1.3b) by

$$E_z(t,z - vt) = \cos\theta\, e^{-i\omega t} \int_{-\infty}^{\infty} \tilde{E}_z(k,\omega)\, e^{ik(z - vt)}\, dk , \qquad (2.3)$$

where we have explicitly introduced both the theta-dependent and the frequency-dependent terms in the definitions to simplify the subsequent expressions for the transformed field components, all of which occur in this problem. We have E_r , E_z , and B_θ proportional to $\cos\theta$, while E_θ , B_r , and B_z are proportional to $\sin\theta$.

From Maxwell's equation and Ohm's law we obtain relationships between the various components. Inside the pipe $(r < b)$ we have

$$\nu^2 \tilde{E}_r = ik \frac{\partial\tilde{E}_z}{\partial r} + ik \frac{(\beta+\beta_w)}{r} \tilde{B}_z - 4\pi k^2 \beta_w(\beta+\beta_w)\, \xi\, n_0 \tilde{f}\, H(a - r) , \qquad (2.4a)$$

$$\nu^2 \tilde{B}_\theta = ik\, (\beta+\beta_w) \frac{\partial\tilde{E}_z}{\partial r} + \frac{ik}{r} \tilde{B}_z - 4\pi k^2 \beta_w\, \xi\, n_0\, \tilde{f}\, H(a - r), \qquad (2.4b)$$

and

$$\nu^2 \tilde{E}_\theta = -\frac{ik}{r}\tilde{E}_z - ik(\beta+\beta_w)\frac{\partial\tilde{B}_z}{\partial r} + 4\pi\, k^2\, \beta_w(\beta+\beta_w)\, \xi\, n_0 \tilde{f}\, H(a-r), \qquad (2.5a)$$

$$\nu^2 \tilde{B}_r = \frac{ik}{r}(\beta+\beta_w)\tilde{E}_z + ik\frac{\partial\tilde{B}_z}{\partial r} - 4\pi\, k^2\, \beta_w\, \xi\, n_0\, \tilde{f}\, H(a-r), \qquad (2.5b)$$

where $\beta_w = \omega/kc$ and $\nu^2 = -k^2[1-(\beta+\beta_w)^2]$. Inside the metal $(r > b)$ we have

$$\lambda^2 \tilde{E}_r = \frac{ik}{r}\frac{\partial\tilde{E}_z}{\partial r} + \frac{ik}{r}(\beta+\beta_w)\tilde{B}_z\ , \qquad (2.6a)$$

$$\lambda^2 \tilde{B}_\theta = [ik(\beta+\beta_w) - 4\pi\sigma/c]\frac{\partial\tilde{E}_z}{\partial r} + \frac{ik}{r}\tilde{B}_z\ , \qquad (2.6b)$$

$$\lambda^2 \tilde{E}_\theta = -\frac{ik}{r}\tilde{E}_z - ik(\beta+\beta_w)\frac{\partial\tilde{B}_z}{\partial r}\ . \qquad (2.6c)$$

$$\lambda^2 \tilde{B}_r = [ik(\beta+\beta_w) - 4\pi\sigma/c]\frac{\tilde{E}_z}{r} + ik\frac{\partial\tilde{B}_z}{\partial r} \qquad (2.6d)$$

with $\lambda^2 = \nu^2 + (4\pi\, ik\sigma/c)(\beta+\beta_w)$.

By means of Eqs. (2.4), (2.5), and (2.6) we see that we can determine the expressions for the components $\tilde{E}_r$, $\tilde{B}_\theta$, $\tilde{E}_\theta$ and $\tilde{B}_r$ from expressions for the components $\tilde{E}_z$ and $\tilde{B}_z$. The transverse fields found from $\tilde{E}_z$, and the transverse fields found from $\tilde{B}_z$, are two independent solutions to Maxwell's equations.

Inside of the pipe $(r < b)$, the equation for $\tilde{E}_z$ and $\tilde{B}_z$ is

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\left(\nu^2-\frac{1}{r^2}\right)\right]\begin{pmatrix}\tilde{E}_z\\ \tilde{B}_z\end{pmatrix}=0\ ; \tag{2.7}$$

inside the metal $(r > b)$ we have

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\left(\lambda^2-\frac{1}{r^2}\right)\right]\begin{pmatrix}\tilde{E}_z\\ \tilde{B}_z\end{pmatrix}=0\ . \tag{2.8}$$

Equations (2.7) and (2.8) are Bessel's equations. In addition to satisfying Equations (2.4) through (2.8), the transformed expressions must satisfy the proper boundary condition at $r = a$ and $r = b$. At $r = a$ we have $\tilde{B}_z$, $\tilde{E}_z$, $\tilde{E}_\theta$ and $\tilde{B}_r$ all continuous, and

$$\tilde{B}_\theta(r = a^+) - \tilde{B}_\theta(r = a^-) = 4\pi/c \int_{a^-}^{a^+} \tilde{j}_z dr = 4\pi\ \beta\ n_0 \xi \tilde{f}\ , \tag{2.9a}$$

and

$$\tilde{E}_r(r = a^+) - \tilde{E}_r(r = a^-) = 4\pi \int_{a^-}^{a^+} \tilde{\rho}\ dr = 4\pi\ n_0 \xi\ \tilde{f}\ . \tag{2.9b}$$

At $r = b$ we have $\tilde{B}_z$, $\tilde{E}_z$, $\tilde{E}_\theta$, $\tilde{B}_\theta$ and $\tilde{B}_r$ all continuous.

The solutions to Equation (2.7) that are valid inside the pipe are

$$\tilde{E}_z = C_1\ J_1(\nu r) + \phi_1(\nu r)\ H(r - a)\ , \tag{2.10a}$$

and

$$\tilde{B}_z = D_1 J_1(\nu r) + \psi_1(\nu r)\ H(r - a)\ , \tag{2.10b}$$

with ϕ_1 and ψ_1 given by

$$\phi_1(\nu r) = M\,[N_1(\nu r)\,J_1(\nu a) - J_1(\nu r)\,N_1(\nu a)] \quad , \tag{2.11a}$$

and

$$\psi_1(\nu r) = P[N_1(\nu r)\,J_1(\nu a) - J_1(\nu r)\,N_1(\nu a)] \quad , \tag{2.11b}$$

and the phases of ν and λ chosen in such a way that the imaginary parts are always positive. The constants M and P are determined by the boundary conditions at $r = a$. The continuity of E_z and B_z is already included in the definition of ϕ_1 and ψ_1 ; of the remaining boundary conditions two are redundant, and we obtain the following equations:

$$M = (2\pi^2\, a\, n_0 \xi\, \tilde{f} ik)\,(1 - \beta^2 - \beta\,\beta_w) \quad , \tag{2.12a}$$

and

$$P = (2\pi^2\, a\, n_0 \xi\, \tilde{f}\, ik)\,\beta_w \quad . \tag{2.12b}$$

The expressions for $\tilde{E}_z$ and $\tilde{B}_z$ in the metal are obtained from Eq. (2.8) and are given by

$$\tilde{E}_z = [C_1 J_1(\nu b) + \phi_1(\nu b)]\,\frac{H_1^{(1)}(\lambda r)}{H_1^{(1)}(\lambda b)} \quad , \tag{2.13a}$$

and

$$\tilde{B}_z = [D_1 J_1(\nu b) + \psi_1(\nu b)]\,\frac{H_1^{(1)}(\lambda r)}{H_1^{(1)}(\lambda b)} \quad . \tag{2.13b}$$

The continuity of $\tilde{E}_z$ and $\tilde{B}_z$ at $r = b$ are included in Eq. (2.13) and the remaining boundary conditions yield explicit expressions for the constants C_1 and D_1. These expressions are

$$C_1 = \{\mathrm{Num}\}_{C_1} / \{\mathrm{Den}\} \quad , \tag{2.14a}$$

and

$$D_1 = \{\mathrm{Num}\}_{D_1} / \{\mathrm{Den}\} \quad , \tag{2.14b}$$

where

$$\begin{aligned}\{\mathrm{Num}\}_{C_1} = -\Bigg\{ & \frac{1}{b^2}\left[\frac{1}{\nu^2} - \frac{1}{\lambda^2}\right]^2 \frac{\phi(\nu b)}{J(\nu b)} - \frac{4\pi\sigma}{ikc} \frac{H'(\lambda b)\phi(\nu b)}{\lambda H(\lambda b) J(\nu b)} (\beta + \beta_w) \left[\frac{J'(\nu b)}{\nu J(\nu b)} - \frac{H'(\lambda b)}{\lambda H(\lambda b)}\right] \\ & - (\beta + \beta_w)^2 \left[\frac{J'(\nu b)}{\nu J(\nu b)} - \frac{H'(\lambda b)}{\lambda H(\lambda b)}\right] \left[\frac{\phi'(\nu b)}{\nu J(\nu b)} - \frac{H'(\lambda b)\phi(\nu b)}{\lambda H(\lambda b) J(\nu b)}\right] \\ & + \frac{(\beta + \beta_w)}{b J(\nu b)} \left[\frac{1}{\nu^2} - \frac{1}{\lambda^2}\right] \left[\frac{J(\nu b)\ \psi'(\nu b)}{\nu J(\nu b)} - \frac{J'(\nu b)\ \psi(\nu b)}{\nu J(\nu b)}\right] \Bigg\} \quad ,\end{aligned} \tag{2.14c}$$

$$\begin{aligned}\{\mathrm{Num}\}_{D_1} = -\Bigg\{ & \frac{1}{b^2}\left[\frac{1}{\nu^2} - \frac{1}{\lambda^2}\right]^2 \frac{\psi(\nu b)}{J(\nu b)} - \frac{4\pi\sigma}{ikc} \frac{H'(\lambda b)}{\lambda H(\lambda b)} (\beta + \beta_w) \left[\frac{\psi'(\nu b)}{\nu J(\nu b)} - \frac{H'(\lambda b)\psi(\nu b)}{\lambda H(\lambda b) J(\nu b)}\right] \\ & - (\beta + \beta_w)^2 \left[\frac{J'(\nu b)}{\nu J(\nu b)} - \frac{H'(\lambda b)}{\lambda H(\lambda b)}\right] \left[\frac{\psi'(\nu b)}{\nu J(\nu b)} - \frac{H'(\lambda b)\psi(\nu b)}{\lambda H(\lambda b) J(\nu b)}\right] , \\ & + \frac{(\beta + \beta_w)}{b J(\nu b)} \left[\frac{1}{\nu^2} - \frac{1}{\lambda^2}\right] \left[\frac{J(\nu b)\phi'(\nu b)}{\nu J(\nu b)} - \frac{J'(\nu b)\phi(\nu b)}{\nu J(\nu b)}\right] \Bigg\} \quad ,\end{aligned} \tag{2.14d}$$

and

$$\{\mathrm{Den}\} = \left\{ \frac{1}{b^2} \left[\frac{1}{\nu^2} - \frac{1}{\lambda^2}\right]^2 - \frac{4\pi\sigma}{ikc} \frac{H'(\lambda b)}{\lambda H(\lambda b)} (\beta + \beta_w) \left[\frac{J'(\nu b)}{\nu J(\nu b)} - \frac{H'(\lambda b)}{\lambda H(\lambda b)}\right] \right.$$

$$\left. - (\beta + \beta_w)^2 \left[\frac{J'(\nu b)}{\nu J(\nu b)} - \frac{H'(\lambda b)}{\lambda H(\lambda b)}\right]^2 \right\} . \qquad (2.14e)$$

In Eqs. (2.14) the subscripts and superscripts on the Bessel, Neumann, and Hankel functions have been omitted for brevity. The prime denotes differentiation with respect to argument.

2. Approximations

The expressions for C_1 and D_1 are exact; we now restrict ourselves to values of ω that are considerably below the cut-off frequency for the pipe (i.e., frequencies such that $\gamma\omega b << c$). We will also assume that the conductivity is high, so that the displacement current in the metal can be neglected.

With these restrictions, which are easily fulfilled in an actual accelerator or storage ring, the expression for λ^2 reduces to

$$\lambda^2 = R \frac{(\beta + \beta_w)}{\beta} ik , \qquad (2.15)$$

with $R = 4\pi\beta\sigma/c$. Next we expand the expression for C_1 and D_1 to first order in the quantity $\sigma^{-1/2}$ with the result

$$C_1 = - \frac{\lambda H(\lambda b)}{H'(\lambda b)} \frac{2ik\beta}{\pi\nu^2 bR} \left\{ (\beta + \beta_w) \frac{M}{J(\nu b)} - \frac{P}{\nu b J'(\nu b)} \right\} , \qquad (2.16a)$$

and

$$D_1 = -\frac{2P}{\pi\nu b J'(\nu b)}\left\{1 + \frac{\nu J(\nu b)H'(\lambda b)}{\lambda J'(\nu b)H(\lambda b)}\right\} + \frac{\lambda H(\lambda b)}{H'(\lambda b)}\ \frac{2\ i\ k\ \beta\ J(\nu b)}{(\beta + \beta_w)\pi\ \nu^3 b^2 R J'(\nu b)}$$

$$\times \left\{(\beta + \beta_w)\ \frac{M}{J(\nu b)} - \frac{P}{\nu b J'(\nu b)}\right\} . \tag{2.16b}$$

When Eqs. (2.16) are substituted into Eqs. (2.10) one obtains expressions for $\tilde{E}_z$ and $\tilde{B}_z$ inside the pipe. Equations (2.4) and (2.5) may then be used to obtain the expressions for the other components. These expressions for the transformed field components may be inverted by means of Eq. (2.3).

If the value of $(z - vt)$ is much larger than either the radius of the pipe or the bunch length, the major contribution to the integral arises (Appendix A - Case 2) for such values of k that $k\,b \ll 1$. As in the preceding section, we again replace $\tilde{f}(k)$ by $\tilde{f}(0) = 1/2\pi$. The region of most physical interest is that in which the observation distance, $(z - vt)$, is large compared with the pipe radius and small compared to the quantity Rb^2 (which is of the order of 10^9cm for a copper pipe of 3-cm radius). Restricting ourselves to the range

$$b \ll |z - vt| \ll Rb^2 \quad , \tag{2.17}$$

we may, by Appendix A - Case 2 and Case 3 , take $k \gg R^{-1}\, b^{-2}$. Thus we have $\nu\, b \ll 1$ (we have already assumed $\gamma\omega b/c \ll 1$) and $\lambda b \gg 1$, so that the expressions for the transformed fields become:

$$\tilde{E}_z = -i \frac{eN\xi(b^2 - a^2)}{\pi a^2 b^2}\left(\frac{k}{\gamma^2} - \frac{\omega\beta}{c}\right) r - \frac{2eN\xi\beta^2}{\pi(2R)^{\frac{1}{2}} b^3} r \left[1 - i \text{ Sign}(K)\right] |K|^{1/2} \tag{2.18a}$$

$$\tilde{E}_r = -\frac{eN\xi(b^2 - a^2)}{\pi a^2 b^2} + \frac{eN\xi\beta^2}{2\pi(2R)^{\frac{1}{2}} b^3}\left\{\left(i \frac{\omega}{\beta c}\right)(3b^2 - r^2)\left[1 - i \text{ Sign}(K)\right] |K|^{1/2} + (b^2 + r^2)\left[1 + i \text{ Sign }(K)\right] |K|^{3/2}\right\} \tag{2.18b}$$

$$\tilde{E}_\theta = \frac{eN\xi(b^2 - a^2)}{\pi a^2 b^2} - \frac{eN\xi\beta^2}{2\pi(2R)^{\frac{1}{2}} b^3}\left\{\left(i \frac{\omega}{\beta c}\right)(3b^2 + r^2)\left[1 - i \text{ Sign}(K)\right] |K|^{1/2} + (b^2 - r^2)\left[1 + i \text{ Sign }(K)\right] |K|^{3/2}\right\} \tag{2.18c}$$

$$\tilde{B}_z = -i \frac{eN\xi(b^2 + a^2)}{\pi a^2 b^2} \frac{\omega}{c} r + \frac{2eN\xi\beta}{\pi(2R)^{\frac{1}{2}} b^3} r\left\{\left(i \frac{\omega}{\beta c}\right)\left[1 + i \text{ Sign}(K)\right] |K|^{-1/2} + \left[1 - i \text{ Sign}(K)\right] |K|^{1/2}\right\} \tag{2.18d}$$

$$\tilde{B}_r = -\frac{eN\xi\beta(b^2 - a^2)}{\pi a^2 b^2} - \frac{2eN\xi\beta}{\pi(2R)^{\frac{1}{2}} b^3}\left[1 + i \text{ Sign}(K)\right] |K|^{-1/2} \tag{2.18e}$$

$$\tilde{B}_\theta = \tilde{B}_r \tag{2.18f}$$

where $K = \frac{1}{\beta} k(\beta + \beta_w)$. We make the change of variables from k to K ; as indicated in Appendix A , this gives all of the field components a phase factor $\exp[-i(z - vt)(\omega/\beta c)]$. Taking into account the additional phase factor $\exp(-i\omega t)$ that occurs in Eq. (2.3), we find the total phase of the field components to be $-\omega z/v$. Thus the phase of the field is a function only of position, and does not vary with time. By means of Table I (Appendix A) Eqs. (2.18) may be inverted, and the resulting expressions for the field components are[8]

$$E_z = \frac{2\, e\, N\, \xi\, \beta^2}{(\pi R)^{\frac{1}{2}} b^3}\; r\; \frac{S(z,t)}{|z - vt|^{3/2}}\; e^{-i\,\omega z/\beta c} \cos\theta \;, \tag{2.19a}$$

$$E_r = -\frac{eN\,\xi\,\beta^2}{2(\pi R)^{\frac{1}{2}} b^3}\, S(z,t) \left\{ \left(i\,\frac{\omega}{\beta c}\right) \frac{(3b^2 - r^2)}{|z - vt|^{3/2}} + \frac{3(b^2 + r^2)}{2|z - vt|^{5/2}} \right\} e^{-i\omega z/\beta c} \cos\theta \tag{2.19b}$$

$$E_\theta = \frac{eN\,\xi\,\beta^2}{2(\pi R)^{\frac{1}{2}} b^3}\, S(z,t) \left\{ \left(i\,\frac{\omega}{\beta c}\right) \frac{(3b^2 + r^2)}{|z - vt|^{3/2}} + \frac{3}{2}\, \frac{(b^2 - r^2)}{|z - vt|^{5/2}} \right\} e^{-i\omega z/\beta c} \sin\theta \tag{2.19c}$$

$$\tilde{B}_z = -\frac{2eN\xi\beta}{(\pi R)^{\frac{1}{2}} b^3}\, r\, S(z,t) \left\{ \frac{1}{|z - vt|^{3/2}} - 2\left(\frac{i\omega}{\beta c}\right) \frac{1}{|z - vt|^{1/2}} \right\} e^{-i\omega z/\beta c} \sin\theta \tag{2.19d}$$

$$B_r = -\frac{4eN\xi\beta}{(\pi R)^{\frac{1}{2}} b^3}\; \frac{S(z,t)}{|z - vt|^{1/2}}\; e^{-i\,\omega z/\beta c} \sin\theta \tag{2.19e}$$

$$B_\theta = -\frac{4eN\xi\beta}{(\pi R)^{\frac{1}{2}} b^3}\; \frac{S(z,t)}{|z - vt|^{1/2}}\; e^{-i\,\omega z/\beta c} \cos\theta \;, \tag{2.19f}$$

where N is the total number of particles and $S(z,t)$ is defined (as before) as unity for $z < vt$ and zero for $z > vt$. Obviously, Eq. (2.19) contains only dominant terms for each field; the fields are not zero when $S(z,t)$ is zero. If we ignore the field components that fall off faster than $|z - vt|^{-1/2}$, we see that a particle moving in the z direction with velocity v and arriving at position z at time t would experience a force in the x direction given by

$$F_x = \frac{4\, e^2 N\, \xi\, \beta^2}{(\pi R)^{\frac{1}{2}}\, b^3} \quad \frac{S(z,t)}{|z - vt|^{\frac{1}{2}}} \quad e^{-i\omega z/\beta c} \quad . \tag{2.20}$$

It will be recalled [from Eq. (2.2)] that the pulse source passed the position z at time z/v and had--at that moment--phase $-(\omega z/v)$.

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APPENDIX A. ESTIMATION OF FOURIER TRANSFORMS

This appendix is devoted to mathematical questions related to the approximate evaluation of Fourier transforms, that is, to the properties of the integral:

$$f(s) = \int_{-\infty}^{\infty} \tilde{f}(k)\ e^{iks}\ dk \ . \qquad (A.1)$$

The general procedure that we employ is to approximate $\tilde{f}(k)$ by a suitable $\tilde{f}_a(k)$ for which the integral in Eq. (A.1) can be evaluated exactly. In the situations encountered in this paper, the $\tilde{f}_a(k)$ are generally piece-wise analytic functions for which the Fourier transforms, although well-known, are nevertheless uncommon. In Table I we list all of the transforms which we require; the results quoted are established in the literature.[9]

A particularly simple transformation can be employed when the nonanalyticity of $\tilde{f}(k)$ [which for the $\tilde{f}(k)$ of Table I is always at $k = 0$] occurs at $k = k_0$. Letting $k = k_0 + K$ we find that Eq. (A.1) becomes

$$f(s) = e^{ik_0 s} \int_{-\infty}^{\infty} \tilde{f}(k_0 + K)\ e^{iKs}\ dK \qquad (A.2)$$

so that the nonanalyticity is transformed to $K = 0$, and the results of Table I are readily applicable.

We turn now to the problem of bounding the error in $f(s)$ generated by replacing $\tilde{f}(k)$ with $\tilde{f}_a(k)$. To state the problem

TABLE I. Various Fourier transforms.

$\tilde{f}(k)$	$f(s)$
$\|k\|^{\frac{1}{2}}$	$-\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\|s\|^{3/2}}$
$-i\ \|k\|^{\frac{1}{2}}\ \mathrm{Sign}\ (k)$	$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\mathrm{Sign}(s)}{\|s\|^{3/2}}$
$\|k\|^{-\frac{1}{2}}$	$2\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\|s\|^{\frac{1}{2}}}$
$i\ \|k\|^{-\frac{1}{2}}\ \mathrm{Sign}\ (k)$	$-2\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\mathrm{Sign}\ (s)}{\|s\|^{\frac{1}{2}}}$
$\|k\|^{3/2}$	$-3/2\ \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{\|s\|^{5/2}}$
$i\ \|k\|^{3/2}\ \mathrm{Sign}\ (k)$	$3/2\ \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\mathrm{Sign}\ (s)}{\|s\|^{5/2}}$
$i\ k\ \ell n\ \|k\|$	$\pi\ \frac{\mathrm{Sign}\ (s)}{s^2}$
$k\ \mathrm{Sign}\ (k)$	$-\ \frac{2}{s^2}$

more positively: $\tilde{f}(k)$ is generally rather complicated; we want to know how roughly we can approximate it without significant loss of accuracy in $f(s)$. There are a number of cases which we must consider.

1. Asymptotic evaluation

The problem of approximately evaluating $f(s)$ in the limit of very large s, is treated exhaustively in Ref. 9. The situation is that the asymptotic behavior of $f(s)$ is determined by the points of nonanalyticity of $\tilde{f}(k)$. Assuming, as is always the case in this report, that $\tilde{f}(k)$ is analytic for $k \neq 0$, we conclude that $\tilde{f}(k)$ may be approximated by $\tilde{f}_a(k)$, with no error in $f(s)$ as $s \to \infty$, provided $\tilde{f}_a(k)$ is analytic for $k \neq 0$ and the singularity in $\tilde{f}_a(k)$ (at $k = 0$) is the same as that in $\tilde{f}(k)$.

2. Evaluation for large argument

We often have the situation that $\tilde{f}(k)$ is well approximated by $\tilde{f}_a(k)$ for $k < k_{01}$, and both $\tilde{f}$ and $\tilde{f}_a$ are analytic except at $k = 0$. It then is true--as an extension of Case 1--that $f(s)$ is well approximated by the transform of $\tilde{f}_a(k)$ provided $s \gg {k_{01}}^{-1}$.

In this paper, where we are concerned with distances large compared to the pipe's transverse dimensions and the bunch length, we will invoke the present theorem to always limit attention to $k \ll k_{01} \approx L^{-1}, a^{-1}, b^{-1}, d^{-1}$, and thus are permitted many simplifying approximations in $\tilde{f}(k)$.

3. Small region of inaccuracy

Suppose $\tilde{f}(k)$ is closely approximated by $\tilde{f}_a(k)$, except for $|k| < k_{02}$ and

$$\int_{-k_{02}}^{k_{02}} \left[|\tilde{f}(k)| - |\tilde{f}_a(k)| \right] \, dk < \delta \quad . \tag{A.3}$$

It is then the case that the transform of $\tilde{f}_a(k)$ differs from $f(s)$ by less than δ .

Consequently, $f(s)$ is well approximated except when it is smaller in value than δ . (This is the reason Case 1 isn't contradicted by the present result, since $f(s)$ generally approaches zero asymptotically.) In our applications, the range k_{02} will be exceedingly small [of the order of $R^{-1}\, b^{-2}$ or $R^{-1}(d - b)^{-2}$, with $R \approx 10^8 \text{ cm}^{-1}$] . It follows that even a rather large departure of $\tilde{f}_a(k)$ from $\tilde{f}(k)$ can be tolerated within k_{02} , with $f(s)$ well approximated except where it is exceedingly small, namely, at very large distances.

APPENDIX B. BESSEL FUNCTION PROPERTIES

In this Appendix we summarize--without derivation--various properties of Bessel functions which are necessary to the analysis employed in the paper. More complete discussions can be found in any standard text.[10]

The Bessel function $J_\nu(x)$, where ν is any integer, is the solution of Bessel's equation:

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dz_\nu}{dx}\right) + \left(1 - \frac{\nu^2}{x^2}\right) z_\nu = 0 , \tag{B.1}$$

defined by the power series

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\nu}}{\Gamma(k+1)\,\Gamma(k+\nu+1)} . \tag{B.2}$$

The Neumann function $N_\nu(x)$ also satisfies Eq. (B.1), but with different boundary conditions, and has the property

$$N_0(x) \approx -\frac{2}{\pi}\,\ell n\,\frac{2}{\gamma x} ,$$

$$N_\nu(x) \approx -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu , \text{ for } x \ll 1 \text{ and } \nu \neq 0 , \tag{B.3}$$

where the (Euler) constant $\gamma = 1.7811$. The Hankel functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are defined by

$$H_\nu^{\binom{1}{2}}(x) = J_\nu(x) \pm i\,N_\nu(x) , \tag{B.4}$$

and clearly also satisfy Eq. (B.1). For small argument, $H_\nu^{\binom{1}{2}}(x)$ may be obtained from Eqs. (B.2) and (B.3). The special case of

$$1 - i\,\frac{\pi}{2}\,x\,H_1^{(1)}(x) = 1 - \frac{i\pi}{2}\,x\,[J_1(x) + i\,N_1(x)]$$

$$\approx 1 + \frac{\pi}{2}\,x\,N_1(x)\;,\quad \text{for} \quad x \ll 1\;,$$

$$\approx \frac{x^2}{2}\,\ell n\,x\;,\quad \text{for} \quad x \ll 1 \tag{B.5}$$

requires the next term in the expansion of Eq. (B.3).

Asymptotically, namely for large argument,

$$J_\nu(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{\pi}{4} - \frac{\pi\nu}{2}\right)\;,$$

$$N_\nu(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin\left(x - \frac{\pi}{4} - \frac{\pi\nu}{2}\right)\;, \tag{B.6}$$

from which it is evident that $H_\nu^{\binom{1}{2}}(ix)$ approaches asymptotically

$$H_\nu^{\binom{1}{2}}(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{\pm i(x - \frac{\pi}{4} - \frac{\pi\nu}{2})} \;. \tag{B.7}$$

The Wronskian relation (the prime indicates derivations with respect to argument)

$$N'_\nu(x)\,J_\nu(x) - J'_\nu(x)\,N_\nu(x) = \frac{2}{\pi x}\;, \tag{B.8}$$

or the equivalent relationship

$$N_{\nu-1}(x)\, J_\nu(x) - J_{\nu-1}(x)\, N_\nu(x) = \frac{2}{\pi x} \qquad \text{(B.9)}$$

is often useful, as are the analogous relations for $H_\nu^{\binom{1}{2}}$:

$$J_{\nu-1}(x)\, H_\nu^{(1)}(x) - J_\nu(x)\, H_{\nu-1}^{(1)}(x) = \frac{2}{\pi i x}$$

$$H_{\nu-1}^{(2)}(x)\, J_\nu(x) - H_\nu^{(2)}(x)\, J_{\nu-1}(x) = \frac{2}{\pi i x} \; . \qquad \text{(B.10)}$$

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